

Coset Sum: an alternative to the tensor product in wavelet construction

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Abstract—A multivariate biorthogonal wavelet system can be obtained from a pair of biorthogonal multivariate refinement masks in Multiresolution Analysis setup. Some multivariate refinement masks may be decomposed into lower dimensional refinement masks. Although tensor product is a popular way to construct a decomposable multivariate refinement mask from lower dimensional refinement masks, it may not be the only method that can achieve this.

We present an alternative method, which we call coset sum, for constructing multivariate refinement masks from univariate refinement masks. The coset sum shares many essential features of the tensor product that make it attractive in practice: (1) it preserves the biorthogonality of univariate refinement masks, (2) it preserves the accuracy number of the univariate refinement mask, and (3) the wavelet system associated with it has fast algorithms for computing and inverting the wavelet coefficients. The coset sum can even provide a wavelet system with faster algorithms in certain cases than the tensor product. These features of the coset sum suggest that it is worthwhile to develop and practice alternative methods to the tensor product for constructing multivariate wavelet systems.

Index Terms—Coset sum, fast algorithm, interpolatory mask, refinement mask, tensor product, wavelet mask, wavelet system.

I. INTRODUCTION

One of the most common tools for constructing wavelets is Multiresolution Analysis (MRA) [1]. In MRA, a biorthogonal multivariate wavelet system can be obtained from a pair of biorthogonal multivariate refinement masks. The tensor product has been the prevailing method for deriving a pair of biorthogonal multivariate refinement masks from a pair of biorthogonal univariate refinement masks.

Throughout this paper, we refer to any of the functions involved in wavelet construction (i.e. mother wavelets, wavelet masks, refinable functions, and refinement masks) as a *wavelet function*. We also refer to an operator that maps lower dimensional wavelet functions to higher dimensional wavelet functions as *liftable*. The multi-D wavelet functions that are obtained by the liftable method are referred to as *decomposable*. The multi-D wavelet functions obtained via tensor product are called *tensor product (or separable)* wavelet functions. Since the word “separable” is reserved for the tensor product by the definition in the literature, we use the word “decomposable” to indicate more general case than the tensor product. It should be noted that a “nonseparable” wavelet function only means

it is not a tensor product wavelet function, and it can still be a “decomposable” wavelet function.

In MRA setup, construction of multi-D biorthogonal wavelet systems can be done by two steps: (i) construction of multi-D biorthogonal refinement masks (or refinable functions); (ii) construction of multi-D wavelet masks. To construct a nonseparable multi-D wavelet system, one can try making the refinement masks nonseparable in step (i) or making wavelet masks nonseparable in step (ii). Since, once a pair of multivariate biorthogonal refinement masks are given, the matrix extension problem of finding wavelet masks can always be solved by using Quillen-Suslin theorem (see, for example, [2]), the main effort so far for constructing nonseparable wavelets has been made in step (i).

Although there have been many methods for constructing nonseparable multi-D wavelets [3]–[16], constructing nonseparable multi-D wavelet systems is highly nontrivial. Many of these methods work only for low spatial dimensions (2-D or 3-D) and they cannot be easily extended to other dimensions. Others assume that the wavelets or refinable functions have a special form (e.g. the refinable function has a box spline factor) and cannot be easily generalized to other cases.

We now make a couple of comments regarding the typical approach of constructing MRA-based multi-D biorthogonal wavelet systems mentioned earlier. First, although there have been many attempts to address the step (i), no liftable method that preserves the biorthogonality of the refinement masks has yet been reported other than the tensor product, to the authors’ best knowledge. Second, Quillen-Suslin theorem serves only as a guide since in the process of determining the wavelet masks, some parameters still need to be specified.

Most multi-D wavelet systems that are used in practice nowadays are separable wavelet systems constructed by the tensor product of 1-D wavelet systems. In §II-B we briefly discuss the use of tensor product in constructing biorthogonal wavelet systems. As we can see from there, the tensor product construction of wavelet systems is extremely simple. This is one of the major reasons the tensor product has been so popular in constructing multi-D wavelets in practice. However the separable wavelet systems have limitations: (i) they have a strong directional bias along lines parallel to the coordinate direction, (ii) they are not very local¹, (iii) the associated computational algorithms are not fast enough. Let us elaborate on (iii). For the tensor product wavelet system with k vanishing

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¹One way to measure the localness of a wavelet system is to compute the sum of the volumes of the supports of its mother wavelets (cf. [17], [18]).

TABLE I
COMPARISON BETWEEN TENSOR PRODUCT AND COSET SUM ($n \geq 2$)

Tensor product \mathcal{T}_n (R, \tilde{R} : univariate refinement masks)	Coset sum \mathcal{C}_n (R, \tilde{R} : univariate refinement masks; \tilde{R} : interpolatory)
$\mathcal{T}_n[R]$ can be decomposed into the product of R	$\mathcal{C}_n[R]$ can be decomposed into the sum of R
$\mathcal{T}_n[R]$ is interpolatory iff R is interpolatory	$\mathcal{C}_n[R]$ is interpolatory iff R is interpolatory
$\mathcal{T}_n[R]$ and $\mathcal{T}_n[\tilde{R}]$ are biorthogonal iff R and \tilde{R} are biorthogonal	$\mathcal{C}_n[R]$ and $\mathcal{C}_n[\tilde{R}]$ are biorthogonal iff R and \tilde{R} are biorthogonal
$\mathcal{T}_n[R]$ and R have the same accuracy number	$\mathcal{C}_n[\tilde{R}]$ and \tilde{R} have the same accuracy number
$\mathcal{T}_n[R]$ can be decomposed into nonunivariate refinement masks	$\mathcal{C}_n[R]$ can be decomposed <i>only</i> into univariate refinement masks
Complexity constant in associated wavelet algorithm increases with n	Complexity constant in associated wavelet algorithm is independent of n

moments (e.g. Daubechies wavelet system of order k [19]), the associated algorithm has complexity $CknN$ (cf. §II-B and [20]), where n is the spatial dimension, N is the size of an initial data to be analyzed, and C is a constant independent of k, n , and N . Thus, the constant in the complexity bound (cf. Complexity discussion in §IV-B for the definition) in this case is Ckn and it grows linearly with the spatial dimension. This is not desirable for high spatial dimensions (i.e. for large n), since typically the size of dataset N increases as n increases and one wants to make the complexity constant as small as possible. We show in §IV-B, in certain cases, this can be improved by using an alternative method to the tensor product.

Our goal in this paper is to present an alternative method to the tensor product for constructing decomposable multi-D refinement masks. We call the new method as *coset sum*. We show that, under an appropriate circumstance, the coset sum shares many attractive features of the tensor product. First, it preserves the biorthogonality of univariate refinement masks. Second, it preserves the accuracy number of the univariate refinement mask. Third, it has a corresponding wavelet system which has fast algorithms for computing and inverting the wavelet coefficients. In fact, it turns out that these algorithms are faster, in certain cases, than the known algorithms based on tensor product wavelet systems (cf. Example 6). The main difference between the two methods is that a “sum” is used in obtaining the coset sum multi-D refinement masks instead of a “product” used in the tensor product refinement masks. Another difference is that, on the contrary to the tensor product case, the coset sum refinement mask cannot be decomposed into non-univariate refinement masks. Table I summarizes the comparison between the tensor product and the coset sum.

The rest of the paper is organized as follows. In §II we briefly overview some relevant concepts on wavelet construction. In §III we introduce the coset sum method and discuss its properties. In §IV we present the wavelet system associated with the coset sum, together with its fast algorithms. We summarize our results and present some observations in §V. All the proofs of the theorems in this paper are included in the Appendix.

II. PRELIMINARIES

In this section we review some relevant concepts.

A. Refinement masks and wavelet masks

In this paper we refer to a Laurent trigonometric polynomial as a *mask*, and a mask τ with $\tau(0) = 1$ as a *refinement mask*. Refinement masks can be used to obtain refinable functions (see, for example, [21]), which can in turn be used to construct wavelet systems [1].

Refinement masks τ and τ^d are *biorthogonal* if they satisfy the following biorthogonal relation:

$$\sum_{\gamma \in \pi\Gamma} (\overline{\tau\tau^d})(\omega + \gamma) = 1, \quad \forall \omega \in \mathbb{T}^n := [-\pi, \pi]^n, \quad (1)$$

where $\Gamma := \{0, 1\}^n$ and the overline is used to denote the complex conjugate.

A refinement mask τ is *interpolatory* if the condition

$$\sum_{\gamma \in \pi\Gamma} \tau(\omega + \gamma) = 1$$

holds. Thus refinement masks τ and τ^d are biorthogonal if and only if $\overline{\tau\tau^d}$ is interpolatory. Interpolatory masks are widely used in subdivision schemes and wavelet constructions (for example, see [22] and references therein).

In this paper we say that a filter $h : \mathbb{Z}^n \rightarrow \mathbb{R}$ is *associated with a mask* τ if h and τ are connected via the relation $\tau(\omega) = \frac{1}{2^n} \sum_{k \in \mathbb{Z}^n} h(k) e^{-ik \cdot \omega}$ for $\omega \in \mathbb{T}^n$.

It is straightforward to see that τ is interpolatory if and only if the associated filter h satisfies

$$h(k) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \in 2\mathbb{Z}^n \setminus 0, \end{cases} \quad (2)$$

to which we refer as the interpolatory condition for the filter.

For a refinement mask τ , the number of zeros of τ at $\gamma \in \pi\Gamma'$ with $\Gamma' := \Gamma \setminus 0 = \{0, 1\}^n \setminus 0$ is referred to as the *accuracy number* [23]. Throughout the paper we assume that all refinement masks have at least one accuracy number, since almost all of the refinement masks used in practice satisfy this condition.

We recall that the Laurant polynomials $\{t_j, t_j^d : j = 1, \dots, l\}$ are called the *wavelet masks* associated with a pair of biorthogonal refinement masks (τ, τ^d) if they satisfy the Mixed Unitary Extension Principle (MUEP) conditions [24]:

for every $\omega \in \mathbb{T}^n$,

$$\overline{\tau(\omega + \gamma)}\tau^d(\omega) + \sum_{j=1}^l \overline{t_j(\omega + \gamma)}t_j^d(\omega) = \begin{cases} 1, & \text{if } \gamma = 0, \\ 0, & \text{if } \gamma \in \pi\Gamma'. \end{cases} \quad (3)$$

When $l = 2^n - 1$, the masks that satisfy the MUEP conditions can be used to construct biorthogonal wavelet systems. We refer to such $(\tau, (t_j)_{j=1, \dots, 2^n-1})$ and $(\tau^d, (t_j^d)_{j=1, \dots, 2^n-1})$ as the *combined biorthogonal masks*. A (MRA-based) *biorthogonal wavelet system* is then obtained from these combined biorthogonal masks, under some simple additional conditions [25], [26].

For a wavelet mask t , the number of zeros of t at $\omega = 0$ is referred to as the *number of (discrete) vanishing moments* [27]. It is well-known (see, for example, [27]) that for the combined biorthogonal masks $(\tau, (t_j)_{j=1, \dots, 2^n-1})$ and $(\tau^d, (t_j^d)_{j=1, \dots, 2^n-1})$ whose refinement masks have at least m accuracy, every wavelet mask t_j and the dual wavelet mask t_j^d , $j = 1, \dots, 2^n - 1$, has at least m vanishing moments. The number of vanishing moments is closely related to the approximation performance of the wavelet system [28].

B. Tensor product wavelet construction

We recall that the n -D tensor product (or separable) refinement mask from n (possibly distinct) univariate refinement masks R_1, R_2, \dots, R_n can be written as, for $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \mathbb{T}^n$,

$$\mathcal{T}_n[R_1, R_2, \dots, R_n](\omega) := R_1(\omega_1)R_2(\omega_2) \cdots R_n(\omega_n). \quad (4)$$

When $R = R_1 = R_2 = \dots = R_n$, we use the notation $\mathcal{T}_n[R]$. If we let H and h be the filters associated with the masks R and $\mathcal{T}_n[R]$ respectively, they satisfy, for $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$,

$$h(k) = H(k_1)H(k_2) \cdots H(k_n).$$

It is well known that the n -D refinement masks constructed using tensor product preserve many useful properties of univariate refinement masks. For example, if we let R and \tilde{R} be univariate refinement masks, then

- (i) $\mathcal{T}_n[R]$ is interpolatory if and only if R is interpolatory,
- (ii) $\mathcal{T}_n[R]$ and $\mathcal{T}_n[\tilde{R}]$ are biorthogonal if and only if R and \tilde{R} are biorthogonal,
- (iii) $\mathcal{T}_n[R]$ and R have the same accuracy number.

Now we pose the following question. Can we find another liftable method that satisfies all of the above properties? An affirmative answer is provided by the coset sum, which we introduce and study in the next section. Before introducing the coset sum, let us review the usual approach for constructing biorthogonal wavelet systems.

Construction of 1-D biorthogonal wavelet systems is well-understood. Given a pair of 1-D biorthogonal refinement masks S_0 and U_0 , one sets the wavelet masks as

$$S_1(\omega) := e^{-i\omega} \overline{U_0(\omega + \pi)}, \quad U_1(\omega) := e^{-i\omega} \overline{S_0(\omega + \pi)} \quad (5)$$

for $\omega \in \mathbb{T}$. Then the univariate pairs (S_0, S_1) and (U_0, U_1) satisfy the MUEP conditions (cf. (3)) [25].

On the other hand, given a pair of multivariate biorthogonal refinement masks, constructing a multivariate biorthogonal wavelet system is not so trivial since one needs to find $2^n - 1$ wavelet masks t_j 's and $2^n - 1$ dual wavelet masks t_j^d 's.

The usual construction of multi-D biorthogonal wavelet systems is done by the tensor product. Given a pair of 1-D biorthogonal refinement masks S_0 and U_0 , one sets the n -D refinement masks as

$$\tau := \mathcal{T}_n[S_0], \quad \tau^d := \mathcal{T}_n[U_0]$$

and the n -D wavelet masks as

$$t_\nu = \mathcal{T}_n[S_{\nu_1}, S_{\nu_2}, \dots, S_{\nu_n}], \quad t_\nu^d = \mathcal{T}_n[U_{\nu_1}, U_{\nu_2}, \dots, U_{\nu_n}]$$

for all $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \Gamma'$. Then $(\tau, (t_\nu)_{\nu \in \Gamma'})$ and $(\tau^d, (t_\nu^d)_{\nu \in \Gamma'})$ satisfy the MUEP conditions (cf. (3)). Here $\Gamma' = \{0, 1\}^n \setminus 0$ is used as before, and the univariate masks S_1 and U_1 are the ones defined in (5). The biorthogonal wavelet systems obtained from these masks are the tensor product (or separable) wavelet systems.

It is well-known that tensor product wavelet systems have fast algorithms for computing and inverting wavelet coefficients (see, for example, [20]), to which we refer as the *fast tensor product wavelet algorithms*. These algorithms have linear complexity $O(N)$, where N is the size of the input data. More precisely, if α is the number of nonzero entries of the filter associated with S_0 and β is the number of nonzero entries of the filter associated with U_0 , then the algorithms for computing and inverting the corresponding tensor product wavelet coefficients have complexity $(\alpha + \beta)nN$, where n is the spatial dimension. In particular, the constant in the complexity bound is $(\alpha + \beta)n$ and it increases as the spatial dimension increases.

III. COSET SUM

A. Introduction to coset sum

We present an alternative method, called *coset sum*, to the tensor product in wavelet construction. Instead of the “product” in the tensor product, we propose to use a “sum” to construct multivariate refinement masks from univariate refinement masks.

Let R be a univariate refinement mask and let H be the univariate filter associated with R . For $\nu \in \Gamma' = \Gamma \setminus 0 = \{0, 1\}^n \setminus 0$, the map

$$\mathbb{T}^n \rightarrow \mathbb{C} : \omega \mapsto \frac{1}{2^{n-1}} R(\omega \cdot \nu),$$

where $\omega \cdot \nu$ is the inner product in \mathbb{R}^n , is an n -D Laurent trigonometric polynomial. The normalization factor $\frac{1}{2^{n-1}}$ is used to place $R(\omega \cdot \nu)$ in the n -D space. In terms of filters, the above can be understood as aligning the 1-D filter H along the ν direction:

$$\mathbb{Z}^n \rightarrow \mathbb{R} : k \mapsto \begin{cases} H(K), & \text{if } k = K\nu \text{ for some } K \in \mathbb{Z}, \\ 0, & \text{otherwise} \end{cases}$$

Since we want to consider all the directions in Γ' , a possible candidate for the coset sum definition can be given as

$$\mathbb{T}^n \rightarrow \mathbb{C} : \omega \mapsto A + \frac{1}{2^{n-1}} \sum_{\nu \in \Gamma'} R(\omega \cdot \nu).$$

Since we want the coset sum to map a 1-D *refinement* mask to an n -D *refinement* mask, by plugging in $\omega = 0$, we obtain $A = -1 + \frac{1}{2^{n-1}}$ and get to the following definition.

Definition 1: We define the coset sum \mathcal{C}_n that maps a 1-D refinement mask R to an n -D refinement mask $\mathcal{C}_n[R]$ as follows: for $\omega \in \mathbb{T}^n$

$$\mathcal{C}_n[R](\omega) := \frac{1}{2^{n-1}} \left(1 - 2^{n-1} + \sum_{\nu \in \Gamma'} R(\omega \cdot \nu) \right),$$

where $\Gamma' = \Gamma \setminus 0 = \{0, 1\}^n \setminus 0$. ■

Remark. We call the refinement mask obtained by the coset sum method as the *coset sum refinement mask*. The set $\Gamma = \{0, 1\}^n$ used in the definition is a complete set of representatives of the distinct cosets (hence the name “coset sum”) of the quotient group $\mathbb{Z}^n / 2\mathbb{Z}^n$. Since the mask R is 2π -periodic, the set $\{0, 1\}^n$ can be replaced by any other complete set of representatives of the distinct cosets of the quotient group $\mathbb{Z}^n / 2\mathbb{Z}^n$. The set $\{0, 1\}^n$ is chosen for the discussion in this paper (with the exception of Example 3) because it makes the support of the associated filter the smallest. Depending on applications, choosing a different set of representatives can make more sense. ■

We note that the coset sum formula in the above definition can be simplified as

$$-1 + \frac{1}{2^{n-1}} \sum_{\nu \in \Gamma'} R(\omega \cdot \nu). \quad (6)$$

The coset sum for the first few low dimensions are given as follows:

$$\mathcal{C}_1[R](\omega) = R(\omega),$$

$$\mathcal{C}_2[R](\omega_1, \omega_2) = \frac{1}{2} \{-1 + R(\omega_1) + R(\omega_2) + R(\omega_1 + \omega_2)\},$$

$$\mathcal{C}_3[R](\omega_1, \omega_2, \omega_3) = \frac{1}{4} \{-3 + R(\omega_1) + R(\omega_2) + R(\omega_1 + \omega_2) + R(\omega_3) + R(\omega_1 + \omega_3) + R(\omega_2 + \omega_3) + R(\omega_1 + \omega_2 + \omega_3)\}.$$

The filter h associated with the coset sum refinement mask $\mathcal{C}_n[R]$ is connected to the univariate filter H via

$$h(k) = \begin{cases} H(K), & \text{if } k = K\nu \text{ for some } K \in \mathbb{Z} \setminus 0, \nu \in \Gamma', \\ 2^n - (2^n - 1)(2 - H(0)), & \text{if } k = 0, \\ 0, & \text{for all other } k \in \mathbb{Z}^n. \end{cases}$$

If the univariate mask R (and the corresponding univariate filter H) is interpolatory, the n -D coset sum refinement mask is also interpolatory and it can be written as

$$\mathcal{C}_n[R] = \frac{1}{2^{n-1}} \left(\frac{1}{2} + \sum_{\nu \in \Gamma'} \left(R(\omega \cdot \nu) - \frac{1}{2} \right) \right), \quad (7)$$

and the associated filter can be expressed as

$$h(k) = \begin{cases} H(K), & \text{if } k = K\nu \text{ for some } K \in \mathbb{Z}, \nu \in \Gamma', \\ 0, & \text{for all other } k \in \mathbb{Z}^n. \end{cases} \quad (8)$$

In particular, the restriction of the n -D filter h to ν direction, for each $\nu \in \Gamma'$, is the 1-D filter H .

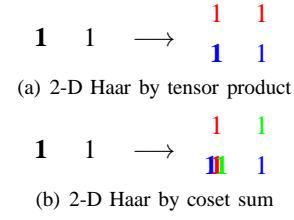


Fig. 1. Constructions of 2-D Haar refinement filter (Tensor product and Coset sum) (cf. Example 1)

Now we give a few very simple examples of constructing multi-D refinement filters from univariate refinement filters.

Example 1: n -D Haar refinement filter: the only filter that can be obtained using either the tensor product or the coset sum. Consider the 2-D Haar refinement filter

$$h(k) = \begin{cases} 1, & \text{if } k = (0, 0), (1, 0), (0, 1) \text{ or } (1, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let H be the 1-D Haar refinement filter

$$H(K) = \begin{cases} 1, & \text{if } K = 0 \text{ or } K = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then h can be obtained from H either by

- (I) (Tensor Product Case) aligning the filter H along $y = 0$ line (x -axis) and $y = 1$ line (see Figure² 1(a)), or by
- (II) (Coset Sum Case) aligning the filter H along $y = 0$ line (x -axis), $x = 0$ line (y -axis), and $y = x$ line (see Figure 1(b)).

Since the support of the 2-D tensor product refinement filter will always be a square and the support of the 2-D coset sum refinement filter will always be the union of three line segments in different directions, it is easy to see that, up to the integer translation, the 2-D Haar refinement filter is the only 2-D filter that can be obtained using either the tensor product or the coset sum. It is straightforward to show that, for arbitrary spatial dimension n , the n -D Haar refinement filter the only filter that can be obtained using either the tensor product or the coset sum. ■

Example 2: Refinement filter associated with an n -D piecewise-linear box spline. Let us consider the 2-D refinement filter h associated with a 2-D piecewise-linear box spline [29]:

$$h(k) = \begin{cases} 1, & \text{if } k = (0, 0), \\ \frac{1}{2}, & \text{if } k = \pm(1, 0), \pm(0, 1), \text{ or } \pm(1, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let H be the refinement filter associated with a 1-D piecewise-linear spline:

$$H(K) = \begin{cases} 1, & \text{if } K = 0, \\ \frac{1}{2}, & \text{if } K = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then h can be obtained from H by aligning the filter H along $y = 0$ line (x -axis), $x = 0$ line (y -axis), and $y = x$ line (see Figure 2). In other words, $h = \mathcal{C}_2[H]$. In fact, it is easy to see

²In the figures of filters drawn in this paper, the bold-faced number is used to represent the value of the filter at the origin.

$$\begin{array}{ccc} \frac{1}{2} & \mathbf{1} & \frac{1}{2} \end{array} \longrightarrow \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}$$

Fig. 2. Construction of 2-D piecewise-linear box spline refinement filter (Coset sum) (cf. Example 2)

$$\begin{array}{ccc} \frac{1}{2} & \mathbf{1} & \frac{1}{2} \end{array} \longrightarrow \begin{array}{cccccc} 0 & 0 & 0 & \frac{1}{2} & 0 & \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \\ 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \end{array}$$

Fig. 3. 2-D coset sum refinement filter supported on different directed line segments (cf. Example 3)

that for the n -D refinement filter h associated with an n -D piecewise-linear box spline, we have $h = \mathcal{C}_n[H]$. ■

Example 3: Refinement filter supported on different directed line segments. We consider $n = 2$ and choose the same univariate filter H as in Example 2, but choose the 2-D filter h differently:

$$h(k) = \begin{cases} 1, & \text{if } k = (0, 0), \\ \frac{1}{2}, & \text{if } k = \pm(1, 2), \pm(2, 1), \text{ or } \pm(-1, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Then $h = \mathcal{C}_2[H]$ with Γ chosen differently (cf. Remark after Definition 1):

$$\Gamma = \{(0, 0), (2, 1), (1, 2), (-1, 1)\}.$$

In particular, h can be obtained from H aligning the filter H along $y = x/2$ line, $y = 2x$ line, and $y = -x$ line (see Figure 3). Note that the filter h is supported on the line segments that are not parallel to the coordinate directions. ■

B. Properties of coset sum refinement masks

In this subsection, we study the properties of the multi-D refinement masks obtained by the coset sum method.

The following theorem shows that the refinement masks obtained by the coset sum share many important properties with the tensor product refinement masks.

Theorem 1: Let \mathcal{C}_n be the coset sum, and let R and \tilde{R} be univariate refinement masks.

- (a) $\mathcal{C}_n[R]$ is interpolatory if and only if R is interpolatory.
- (b) Suppose that one of R and \tilde{R} is interpolatory. Then $\mathcal{C}_n[R]$ and $\mathcal{C}_n[\tilde{R}]$ are biorthogonal if and only if R and \tilde{R} are biorthogonal.
- (c) Suppose that R is interpolatory. Then $\mathcal{C}_n[R]$ and R have the same accuracy number.

Proof: See Appendix A. ■

Below we add a few remarks on Theorem 1.

Remark on Theorem 1(b). The interpolatory condition in part (b) cannot be omitted. To see this, we consider the univariate

$$\begin{array}{ccccccc} -\frac{1}{16} & 0 & \frac{9}{16} & 1 & \frac{9}{16} & 0 & -\frac{1}{16} \\ \text{Filter associated with } U_4 \\ \downarrow \mathcal{C}_2 \text{ (Coset sum)} \\ \begin{array}{ccccccc} 0 & 0 & 0 & -\frac{1}{16} & 0 & 0 & -\frac{1}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{9}{16} & \frac{9}{16} & 0 & 0 \\ -\frac{1}{16} & 0 & \frac{9}{16} & 1 & \frac{9}{16} & 0 & -\frac{1}{16} \\ 0 & 0 & \frac{9}{16} & \frac{9}{16} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{16} & 0 & 0 & -\frac{1}{16} & 0 & 0 & 0 \end{array} \\ \text{Filter associated with } \mathcal{C}_2[U_4] \end{array}$$

Fig. 4. Refinement filters associated with the masks U_4 and $\mathcal{C}_2[U_4]$ in Example 4

refinement mask associated with Daubechies wavelet system of order 2 [19], and let

$$R(\omega) = \tilde{R}(\omega) = \cos^2\left(\frac{\omega}{2}\right) \left(\frac{1 + \sqrt{3}}{2} + \frac{1 - \sqrt{3}}{2} e^{-i\omega} \right), \omega \in \mathbb{T}.$$

Then R (hence \tilde{R}) is not interpolatory, and R and \tilde{R} are biorthogonal. However it is easy to see that $\mathcal{C}_2[R]$ and $\mathcal{C}_2[\tilde{R}]$ are not biorthogonal. ■

Remark on Theorem 1(c). For general (not necessarily interpolatory) R , the accuracy number of $\mathcal{C}_n[R]$ is at least $\min\{m_1, m_2\}$ where m_1 is the accuracy number of R and m_2 is the order that $1 - R$ has a zero at the origin. This statement can be proved using arguments in the proof of Theorem 1(c), and we omit the proof. ■

The Deslauriers-Dubuc mask [30] of order $2k$ ($k \in \mathbb{N}$) is defined as

$$U_{2k}(\omega) := \cos^{2k}\left(\frac{\omega}{2}\right) P_k(\sin^2\left(\frac{\omega}{2}\right)), \quad (9)$$

$$P_k(x) := \sum_{j=0}^{k-1} \frac{(k-1+j)!}{j!(k-1)!} x^j.$$

The mask U_{2k} is interpolatory and has accuracy number $2k$. We now present a family of biorthogonal coset sum refinement masks based on the Deslauriers-Dubuc interpolatory masks.

Example 4: A family of n -D biorthogonal coset sum refinement masks. For each $k \in \mathbb{N}$, we choose U_{2k} in (9) as a univariate interpolatory refinement mask. By Theorem 1(a)(c), $\mathcal{C}_n[U_{2k}]$ is an n -D interpolatory refinement mask with accuracy number $2k$. It is straightforward to see that for each $k \in \mathbb{N}$,

$$S_{2k} := U_{2k}(3 - 2U_{2k}) \quad (10)$$

is biorthogonal³ to U_{2k} . By Theorem 1(b), $\mathcal{C}_n[U_{2k}]$ is biorthogonal to $\mathcal{C}_n[S_{2k}]$. Since S_{2k} has at least $2k$ accuracy and $1 - S_{2k}$ has a zero of order at least $2k$ at the origin, by the Remark on Theorem 1(c), $\mathcal{C}_n[S_{2k}]$ has at least $2k$ accuracy. The filters for the cases $k = n = 2$ are depicted in Figure 4 and 5. ■

³Given a refinement filter, a dual refinement filter is not uniquely determined in general. The specific choice of the dual filter of U_{2k} as in (10) was obtained after scrutinizing a result (Theorem 2) of [31].

$-\frac{2}{512}$	0	$\frac{36}{512}$	$-\frac{32}{512}$	$-\frac{126}{512}$	$\frac{288}{512}$	$\frac{696}{512}$	$\frac{288}{512}$	$-\frac{126}{512}$	$-\frac{32}{512}$	$\frac{36}{512}$	0	$-\frac{2}{512}$
Filter associated with S_4												
$\downarrow \mathcal{C}_2$ (Coset sum)												
0	0	0	0	0	0	$-\frac{2}{512}$	0	0	0	0	0	$-\frac{2}{512}$
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	$\frac{36}{512}$	0	0	0	$\frac{36}{512}$	0	0
0	0	0	0	0	0	$-\frac{32}{512}$	0	0	$-\frac{32}{512}$	0	0	0
0	0	0	0	0	0	$-\frac{126}{512}$	0	$-\frac{126}{512}$	0	0	0	0
0	0	0	0	0	0	$\frac{288}{512}$	$\frac{288}{512}$	0	0	0	0	0
$-\frac{2}{512}$	0	$\frac{36}{512}$	$-\frac{32}{512}$	$-\frac{126}{512}$	$\frac{288}{512}$	$\frac{1064}{512}$	$\frac{288}{512}$	$-\frac{126}{512}$	$-\frac{32}{512}$	$\frac{36}{512}$	0	$-\frac{2}{512}$
0	0	0	0	0	0	$\frac{288}{512}$	$\frac{288}{512}$	0	0	0	0	0
0	0	0	0	0	$-\frac{126}{512}$	0	$-\frac{126}{512}$	0	0	0	0	0
0	0	0	$-\frac{32}{512}$	0	0	$-\frac{32}{512}$	0	0	0	0	0	0
0	0	$\frac{36}{512}$	0	0	0	$\frac{36}{512}$	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
$-\frac{2}{512}$	0	0	0	0	0	$-\frac{2}{512}$	0	0	0	0	0	0
Filter associated with $\mathcal{C}_2[S_4]$												

Fig. 5. Refinement filters associated with the masks S_4 and $\mathcal{C}_2[S_4]$ in Example 4

Similar to the tensor product case, the coset sum can actually take different univariate refinement masks. However, since the cardinality of the set Γ' is $2^n - 1$, we have $2^n - 1$ different directions to consider, instead of n different coordinate directions for the tensor product case. In such a case the n -D coset sum refinement can be written as

$$\mathcal{C}_n[(R_\nu)_{\nu \in \Gamma'}](\omega) := \frac{1}{2^{n-1}} \left(1 - 2^{n-1} + \sum_{\nu \in \Gamma'} R_\nu(\omega \cdot \nu) \right), \quad (11)$$

where R_ν , $\nu \in \Gamma'$, are possibly distinct univariate refinement masks for different direction ν .

Let $n = n_1 + n_2 + \dots + n_m$, $n_j \geq 1$ for $j = 1, 2, \dots, m$. Then the tensor product refinement mask in (4) can be written as the product of possibly non-univariate lower dimensional tensor product refinement masks as follows: for $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \mathbb{T}^n$,

$$\begin{aligned} \mathcal{T}_n[R_1, \dots, R_n](\omega) \\ = \mathcal{T}_{n_1}[R_1, \dots, R_{n_1}](\omega_1, \dots, \omega_{n_1}) \cdot \\ \mathcal{T}_{n_2}[R_{n_1+1}, \dots, R_{n_1+n_2}](\omega_{n_1+1}, \dots, \omega_{n_1+n_2}) \cdot \\ \dots \mathcal{T}_{n_m}[R_{n_1+\dots+n_{m-1}+1}, \dots, R_n](\omega_{n_1+\dots+n_{m-1}+1}, \dots, \omega_n). \end{aligned}$$

On the contrary, the coset sum refinement mask *cannot* be written as the sum of non-univariate lower dimensional coset sum refinement masks.

We can also consider a hybrid of the coset sum and the tensor product : for $n = n_1 + n_2 + \dots + n_m$, $n_j \geq 1$, $j = 1, 2, \dots, m$,

$$\begin{aligned} \mathcal{C}_{n_1}[R](\omega_1, \dots, \omega_{n_1}) \cdot \mathcal{C}_{n_2}[R](\omega_{n_1+1}, \dots, \omega_{n_1+n_2}) \cdot \\ \dots \mathcal{C}_{n_m}[R](\omega_{n_1+\dots+n_{m-1}+1}, \dots, \omega_n). \end{aligned} \quad (12)$$

Similar statements to the ones of Theorem 1 can be made for the coset sum refinement mask in a generalized sense as in (11) and for the hybrid refinement mask as in (12). We omit

Multivariate refinement masks

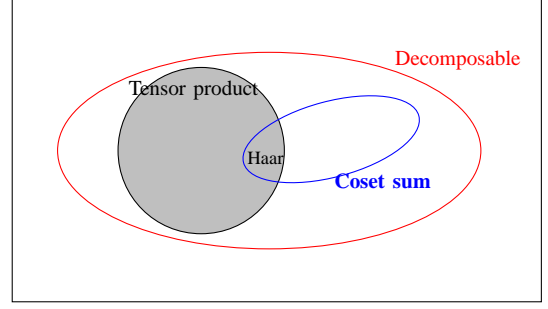


Fig. 6. The tensor product multivariate refinement masks are not the only decomposable refinement masks. The coset sum provides a systematic way to construct other types of decomposable refinement masks. The multivariate Haar refinement mask is essentially the only mask that can be obtained by using either the tensor product or the coset sum (cf. Example 1).

the statements and the proofs as they are similar to the ones of Theorem 1.

The diagram in Figure 6 illustrates the relation among the tensor product, the coset sum, and the decomposable multi-D refinement masks.

IV. APPLICATION: COSET SUM WAVELET SYSTEMS

In this section we introduce a special class of wavelet systems that can be derived from coset sum refinement masks in a very simple manner, and study their fast algorithms.

A. Coset sum wavelet systems

Since the coset sum provides a way to construct a pair of biorthogonal multivariate refinement masks from univariate ones, it can be combined with any procedure for finding wavelet masks to construct a biorthogonal multivariate wavelet system. Below we present a procedure that stands out in terms of simplicity of the construction.

Suppose that S and U are 1-D biorthogonal refinement masks, and that U is interpolatory. Theorem 1(b) implies that the n -D coset sum refinement masks $\mathcal{C}_n[S]$ and $\mathcal{C}_n[U]$ are biorthogonal. Moreover, since U is interpolatory, from (7) and (8) we see that the restriction of the n -D mask $\mathcal{C}_n[U]$ to ν direction, $\nu \in \Gamma' = \{0, 1\}^n \setminus \{0\}$, is given by $U(\omega \cdot \nu)$ for $\omega \in \mathbb{T}^n$, which is essentially a 1-D mask. Hence, as in the 1-D wavelet construction (cf. (5)), one can attempt to define the multivariate wavelet masks t_ν , $\nu \in \Gamma'$, (note that we have $2^n - 1$ wavelet masks) of the form

$$t_\nu(\omega) = e^{-i\omega \cdot \nu} \overline{U(\omega \cdot \nu + \pi)}, \quad \omega \in \mathbb{T}^n. \quad (13)$$

The next theorem shows that the above approach leads to the construction of n -D biorthogonal wavelet systems.

Theorem 2: Suppose that S and U are 1-D biorthogonal refinement masks, and that U is interpolatory. Define n -D biorthogonal refinement masks as

$$\tau := \mathcal{C}_n[S], \quad \tau^d := \mathcal{C}_n[U],$$

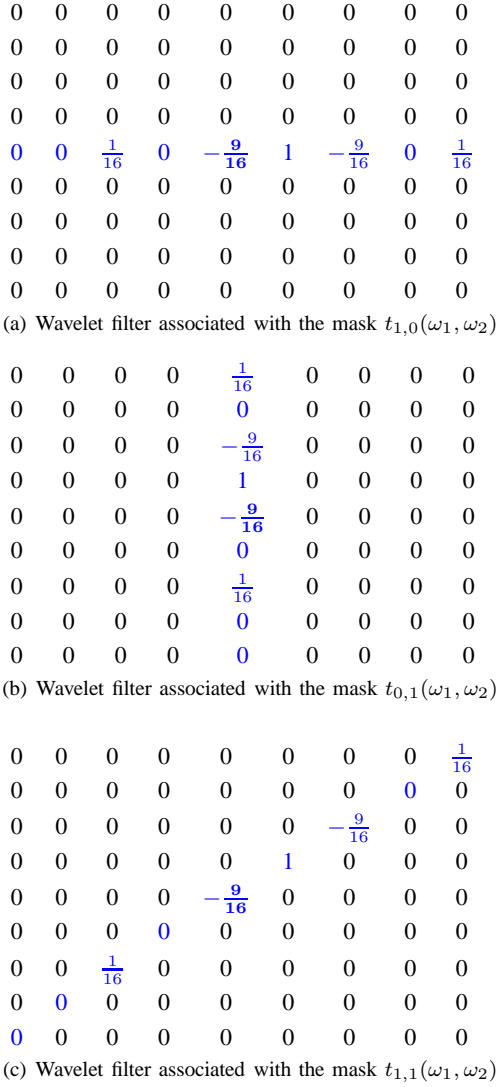


Fig. 7. Coset sum wavelet filters of Example 5 for 2-D with 4 vanishing moments ($n = k = 2$).

and n -D wavelet masks t_ν , $\nu \in \Gamma'$, as in (13). Then there exist dual wavelet masks t_ν^d , $\nu \in \Gamma'$, such that $(\tau, (t_\nu)_{\nu \in \Gamma'})$ and $(\tau^d, (t_\nu^d)_{\nu \in \Gamma'})$ are n -D combined biorthogonal masks. ■

Proof: See Appendix B. ■

Remark. We refer to the biorthogonal wavelet system constructed from the n -D combined biorthogonal masks in Theorem 2 as the *coset sum wavelet system*. ■

Example 5: A family of n -D coset sum wavelet systems. We choose the univariate refinement masks U_{2k} (interpolatory) and S_{2k} as in (9) and (10), respectively, and apply Theorem 2. Then with the wavelet masks given as

$$t_\nu(\omega) = e^{-i\omega \cdot \nu} \sin^{2k}\left(\frac{\omega \cdot \nu}{2}\right) P_k(\cos^2(\frac{\omega \cdot \nu}{2})), \quad \nu \in \Gamma',$$

with $\Gamma' = \{0, 1\}^n \setminus 0$, there exist dual wavelet masks t_ν^d , $\nu \in \Gamma'$, such that $(\mathcal{C}_n[S_{2k}], (t_\nu)_{\nu \in \Gamma'})$ and $(\mathcal{C}_n[U_{2k}], (t_\nu^d)_{\nu \in \Gamma'})$ are n -D combined biorthogonal masks. It is easy to see that all the wavelet masks have $2k$ vanishing moments. All the wavelet filters are supported on the union of $2^n - 1$ line segments

along ν direction for $\nu \in \Gamma'$. For example, if $n = 2$, then $\Gamma' = \{(1, 0), (0, 1), (1, 1)\}$ and the wavelet masks for the case $k = 2$ are given as

$$\begin{aligned} t_{(1,0)}(\omega_1, \omega_2) &= e^{-i\omega_1} \sin^4\left(\frac{\omega_1}{2}\right) \left(1 + 2\cos^2\left(\frac{\omega_1}{2}\right)\right), \\ t_{(0,1)}(\omega_1, \omega_2) &= e^{-i\omega_2} \sin^4\left(\frac{\omega_2}{2}\right) \left(1 + 2\cos^2\left(\frac{\omega_2}{2}\right)\right), \\ t_{(1,1)}(\omega_1, \omega_2) &= e^{-i(\omega_1 + \omega_2)} \sin^4\left(\frac{\omega_1 + \omega_2}{2}\right) \\ &\quad \cdot \left(1 + 2\cos^2\left(\frac{\omega_1 + \omega_2}{2}\right)\right). \end{aligned}$$

The associated wavelet filters are depicted in Figure 7. ■

B. Fast coset sum wavelet algorithms

Next we show that the coset sum wavelet system can be implemented by the fast algorithm with linear complexity whose complexity constant does not grow with the spatial dimension.

Fast Coset Sum Wavelet Algorithms. Let S and U be biorthogonal univariate refinement masks, where U is interpolatory. For the sake of simplicity, we assume that S and U satisfy $S(-\cdot) = S$ and $U(-\cdot) = U$. Let G and H be the filters associated with the refinement masks S and U , respectively.

input $y_J : \mathbb{Z}^n \rightarrow \mathbb{R}$

(1) Decomposition Algorithm:

$$a_G = -2^n + 2 + (2^n - 1)G(0)$$

for $j = J, J-1, \dots, 1$

for $k \in \mathbb{Z}^n$

$$y_{j-1}(k)$$

$$= \frac{1}{2^n} (a_G y_j(2k) + \sum_{\nu \in \Gamma'} \sum_{L \in \mathbb{Z} \setminus 0} G(L) y_j(2k + L\nu)) \quad (\text{i})$$

end

for $\nu \in \Gamma'$ and $k \in \mathbb{Z}^n$

$$w_{\nu, j-1}(k)$$

$$= \frac{1}{2} (y_j(2k + \nu) - \sum_{m \equiv 1} H(m) y_j(2k + (1-m)\nu)) \quad (\text{ii})$$

end

for $k \in \mathbb{Z}^n$

$$A_{j-1}(k) = y_j(2k) - y_{j-1}(k)$$

end

end

(2) Reconstruction Algorithm:

for $j = 1, \dots, J-1, J$

for $k \in \mathbb{Z}^n$

$$y_j(2k) = A_{j-1}(k) + y_{j-1}(k) \quad (\text{iii})$$

end

for $\nu \in \Gamma'$ and $k \in \mathbb{Z}^n$

$$y_j(2k + \nu)$$

$$= 2w_{\nu, j-1}(k) + \sum_{m \equiv 1} H(m) y_j(2k + (1-m)\nu) \quad (\text{iv})$$

end

end

Given coarse coefficients y_j at level j , Decomposition Algorithm first computes the lower level coarse coefficients y_{j-1} . It then computes the wavelet coefficients $w_{\nu, j-1}$, $\nu \in \Gamma' = \{0, 1\}^n \setminus 0$, and auxiliary coefficients A_{j-1} . The auxiliary

coefficients A_{j-1} do not appear in the usual wavelet decomposition algorithm. They are computed here in order to make Reconstruction Algorithm faster (cf. Complexity discussion below). In Step (ii) and (iv), $m \equiv 1$ is used to mean that m is congruent to 1 in modulo 2, i.e., m is an odd integer.

Reconstruction Algorithm recovers y_j from y_{j-1} , $w_{\nu,j-1}$, $\nu \in \Gamma'$, and A_{j-1} . It first recovers y_j at even points (cf. Step (iii)) and then at all other points (cf. Step (iv)). Step (iii) is possible since we stored the auxiliary coefficients A_{j-1} in Decomposition Algorithm. Step (iv) is possible since the only y_j values we need at this step are the values at even points, and these are already computed in Step (iii).

Complexity. We measure complexity by counting the number of operations needed in order to fully derive y_{j-1} , $w_{\nu,j-1}$, $\nu \in \Gamma'$, and A_{j-1} from y_j , and add the number of operations needed for the reconstruction. Here, we count only multiplicative operations such as multiplication and division, as counting additive operations gives a similar result. Thus, for example, computing one entry in Step (i) requires $(2^n - 1)(\alpha - 1) + 1 + n$ operations, where α is the number of nonzero entries of the filter G .

As in the fast tensor product wavelet algorithms discussed in §II-B, the complexity here is linear, i.e. $\sim CN$, with N the number of nonzero entries in y_J , and C some constant independent of y_J . We refer to this constant as the *constant in the complexity bound* or simply as the *complexity constant* throughout this paper.

We now estimate the complexity constant for fast coset sum wavelet algorithms by computing the mean number of operations per single entry in y_J . We observe that the cost per entry of performing one complete cycle of decomposition/reconstruction is bounded by

$$\frac{1}{2^n} \left((2n - 1) + ((2^n - 1)(\alpha - 1) + 1 + n) + 2(2^n - 1)\beta \right) \leq \alpha + 2\beta + 1,$$

where β is the number of nonzero entries of the filter H . Therefore, the algorithm has complexity $(\alpha + 2\beta + 1)N$, and the complexity constant in this case is $\alpha + 2\beta + 1$, which does not increase as the spatial dimension n increases. This is contrary to the tensor product case where the constant grows with the dimension (cf. §II-B). There are a couple of components that make the coset sum wavelet algorithm this fast. First, as we discussed earlier (cf. (13)), the wavelet masks of the coset sum wavelet system are essentially univariate. Second, the reconstruction step can be done by completely bypassing the dual wavelet filters. This is reminiscent of the Laplacian pyramid [32], which has a trivial reconstruction algorithm (cf. Appendix B). Here, it is essential to make the auxiliary coefficients A_{j-1} available in Reconstruction Algorithm by storing them in Decomposition Algorithm. ■

Below we compare the fast tensor product wavelet algorithms with the fast coset sum wavelet algorithms, both obtained from the Deslauriers-Dubuc mask and its dual mask in §III-B.

Example 6: Fast tensor product wavelet algorithms vs. fast coset sum wavelet algorithms. In this example, we compare

the algorithms for two different families of n -D wavelet systems constructed from the same univariate refinement masks by using different liftable methods: (I) the tensor product and (II) the coset sum. We consider the same univariate refinement masks as in Example 4 and 5, i.e. U_{2k} (interpolatory) and S_{2k} as in (9) and (10), respectively. It is easy to see that the number of nonzero entries of the filter associated with S_{2k} is $\alpha = 8k - 3$, and the number of nonzero entries of the filter associated with U_{2k} is $\beta = 2k + 1$.

Then complexity constant for each algorithm is given as follows:

- (I) (Tensor Product Case) From §II-B, the complexity constant for the fast tensor product algorithm is $(\alpha + \beta)n = (10k - 2)n$, which grows with the dimension.
- (II) (Coset Sum Case) From the above Complexity discussion, the complexity constant for the fast coset sum wavelet algorithm is $\alpha + 2\beta + 1 = 8k - 3 + 2(2k + 1) + 1 = 12k$, which *does not* grow with the dimension.

Therefore if we fix k (hence the number of vanishing moments of the wavelet system) and increase the dimension n , then the *complexity constant stays the same for the coset sum case, whereas it increases for the tensor product case.* ■

V. SUMMARY AND OUTLOOK

In this paper we presented the coset sum as an alternative method to the tensor product in constructing decomposable multivariate refinement masks. The decomposable refinement mask constructed by coset sum can be written as the sum, instead of the product, of the univariate refinement masks. We showed that the coset sum can provide many important features of the tensor product, such as preserving the biorthogonality of the univariate refinement masks and the availability of a wavelet system with fast algorithms.

Since the coset sum provides a way to obtain a pair of biorthogonal multivariate refinement masks, it can be combined with any method for finding wavelet masks to construct a (MRA-based biorthogonal) multivariate wavelet system. There has been little progress in a systematic construction of non-tensor based multivariate wavelet systems. The coset sum opens a new opportunity to this end.

The fast tensor product wavelet algorithm has linear complexity, but the constant in the complexity bound increases as the spatial dimension increases. On the other hand, the constant in the (linear) complexity bound for the fast coset sum wavelet algorithm is independent of the dimension. Thus, when the spatial dimension is high, the coset sum wavelet algorithm can be *faster* than the tensor product wavelet algorithm.

Coset sum is not necessarily the only alternative to the tensor product. Rather, its existence with desirable features suggests that it may be worthwhile to develop and practice alternative methods to the tensor product for constructing multivariate wavelet systems.

APPENDIX

A. Proof of Theorem 1

1) *Proof of part (a):* Suppose H and h are the filters associated with masks R and $C_n[R]$. If R is interpolatory, by

(2), $H(0) = 1$, and at all other even points $H(K) = 0$. Then $h(0) = 2^n - (2^n - 1)(2 - H(0)) = 1$ and $h(k) = H(K) = 0$ if $k = K\nu$ for some $K \in 2\mathbb{Z} \setminus 0$, $\nu \in \Gamma'$. Thus $h(0) = 1$ and $h(k) = 0$ if $k \in 2\mathbb{Z} \setminus 0$, i.e., $\mathcal{C}_n[R]$ is also interpolatory.

2) *Proof of part (b):* Without loss of generality, we may assume \tilde{R} is interpolatory. We want to show that, $\mathcal{C}_n[R]$ and $\mathcal{C}_n[\tilde{R}]$ are biorthogonal if and only if R and \tilde{R} are biorthogonal.

Let $R^o := (R - R(\cdot + \pi))/2$ and $R^e := (R + R(\cdot + \pi))/2$ be the odd and even parts of R , respectively, and let \tilde{R}^o be the odd part of \tilde{R} . Since \tilde{R} is interpolatory, the even part of \tilde{R} is the constant $1/2$. It is easy to check $\forall \omega_1 \in \mathbb{T}$

$$\overline{R^o(\omega_1)} \tilde{R}^o(\omega_1) = \frac{1}{2} - \frac{1}{2} \overline{R^e(\omega_1)} \\ \iff R \text{ and } \tilde{R} \text{ are biorthogonal.}$$

Here, as before, the overline is used to denote the complex conjugate.

We will also need the following identities:

$$\sum_{\gamma \in \pi\Gamma} e^{-i\nu \cdot \gamma} = \begin{cases} 2^n, & \text{if } \nu = 0, \\ 0, & \text{if } \nu \in \Gamma'. \end{cases} \quad (14)$$

Then from the definition of the coset sum (cf. Definition 1, (6) and (7)), biorthogonal condition (1), and the above identities (14), we have

$$\begin{aligned} & \mathcal{C}_n[R] \text{ and } \mathcal{C}_n[\tilde{R}] \text{ are biorthogonal} \\ \iff & \sum_{\gamma \in \pi\Gamma} (\overline{\mathcal{C}_n[R]} \mathcal{C}_n[\tilde{R}])(\omega + \gamma) = 1, \quad \forall \omega \in \mathbb{T}^n \\ \iff & \left(\frac{1}{2^{n-1}} \right)^2 \sum_{\gamma \in \pi\Gamma} \left(-2^{n-1} + \sum_{\nu \in \Gamma} \overline{R((\omega + \gamma) \cdot \nu)} \right) \\ & \left(\frac{1}{2} + \sum_{\tilde{\nu} \in \Gamma'} \left(\tilde{R}((\omega + \gamma) \cdot \tilde{\nu}) - \frac{1}{2} \right) \right) = 1, \quad \forall \omega \in \mathbb{T}^n \\ \iff & \sum_{\gamma \in \pi\Gamma} \left(1 - 2^{n-1} + \sum_{\nu \in \Gamma'} e^{i\gamma \cdot \nu} \overline{R^o(\omega \cdot \nu)} + \sum_{\nu \in \Gamma'} \overline{R^e(\omega \cdot \nu)} \right) \\ & \left(\frac{1}{2} + \sum_{\tilde{\nu} \in \Gamma'} e^{-i\gamma \cdot \tilde{\nu}} \tilde{R}^o(\omega \cdot \tilde{\nu}) \right) = (2^{n-1})^2, \quad \forall \omega \in \mathbb{T}^n \\ \iff & 2^{n-1}(1 - 2^{n-1}) + 2^{n-1} \sum_{\nu \in \Gamma'} \overline{R^e(\omega \cdot \nu)} \\ & + 2^n \sum_{\nu \in \Gamma'} \overline{R^o(\omega \cdot \nu)} \tilde{R}^o(\omega \cdot \nu) = (2^{n-1})^2, \quad \forall \omega \in \mathbb{T}^n \\ \iff & \overline{R^o(\omega_1)} \tilde{R}^o(\omega_1) = \frac{1}{2} - \frac{1}{2} \overline{R^e(\omega_1)}, \quad \forall \omega_1 \in \mathbb{T}. \end{aligned}$$

Therefore, $\mathcal{C}_n[R]$ and $\mathcal{C}_n[\tilde{R}]$ are biorthogonal if and only if R and \tilde{R} are biorthogonal.

3) *Proof of part (c):* Let R be a univariate interpolatory refinement mask with accuracy number m . First let us prove the accuracy number of $\mathcal{C}_n[R]$ is at least m . Since R has accuracy number m ,

$$(D^k R)(\pi) = 0, \quad \forall 0 \leq k \leq m-1, \text{ and } (D^m R)(\pi) \neq 0. \quad (15)$$

Furthermore, since R is interpolatory, $1 - R(\omega) = R(\omega + \pi)$ holds for all $\omega \in \mathbb{T}$. Hence $(D^k(1 - R))(0) = (D^k R)(\pi)$ for

all $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Thus $1 - R$ has a zero of order m at the origin, i.e.

$$\begin{aligned} R(0) &= 1 \\ (D^k R)(0) &= 0, \quad \forall 1 \leq k \leq m-1 \\ (D^m R)(0) &\neq 0. \end{aligned} \quad (16)$$

Now consider the n -D refinement mask $\mathcal{C}_n[R]$. The accuracy number of $\mathcal{C}_n[R]$ is at least one, i.e. $\mathcal{C}_n[R](\gamma) = 0$, for all $\gamma \in \pi\Gamma'$. To see this, we need the dual identities of (14):

$$\sum_{\nu \in \Gamma} e^{-i\nu \cdot \gamma} = \begin{cases} 2^n, & \text{if } \gamma = 0, \\ 0, & \text{if } \gamma \in \pi\Gamma'. \end{cases} \quad (17)$$

From (17), we can read off

$$\#\{\nu \in \Gamma' : \gamma \cdot \nu \equiv \pi \pmod{2\pi\mathbb{Z}}\} = 2^{n-1}, \quad (18)$$

for all $\gamma \in \pi\Gamma'$. In particular, the left-hand side of (18) is independent of γ . We then have for any $\gamma \in \pi\Gamma'$

$$\begin{aligned} 2^{n-1} \mathcal{C}_n[R](\gamma) &= -2^{n-1} + \sum_{\nu \in \Gamma} R(\gamma \cdot \nu) \\ &= 1 - 2^{n-1} + \sum_{\{\nu \in \Gamma' : \gamma \cdot \nu \equiv 0\}} R(\gamma \cdot \nu) + \sum_{\{\nu \in \Gamma' : \gamma \cdot \nu \equiv \pi\}} R(\gamma \cdot \nu) \\ &= 0, \end{aligned}$$

where \equiv in the second line is used to denote congruence in modulo $2\pi\mathbb{Z}$, and the last equality is from the conditions $R(0) = 1$, $R(\pi) = 0$ and the identity (18). Furthermore, for all $\gamma \in \pi\Gamma'$ and for all $\mu \in \mathbb{N}_0^n$ with $1 \leq |\mu| \leq m-1$ ($|\mu| := \mu_1 + \dots + \mu_n$)

$$\begin{aligned} (D^\mu \mathcal{C}_n[R])(\gamma) &= \frac{1}{2^{n-1}} \sum_{\nu \in \Gamma'} (D^\mu [R(\omega \cdot \nu)])|_{\omega=\gamma} \\ &= \frac{1}{2^{n-1}} \sum_{\nu \in \Gamma'} \left(\prod_{j=1}^n \nu_j^{\mu_j} \right) (D^{|\mu|} R)(\gamma \cdot \nu) = 0, \end{aligned}$$

where the last equality is from the identities (15) and (16). Therefore the accuracy number of $\mathcal{C}_n[R]$ is at least m .

Next we prove the accuracy number of $\mathcal{C}_n[R]$ is exactly m by contradiction. Suppose the accuracy number of $\mathcal{C}_n[R]$ is $m+l$ with $l \geq 1$. Then

$$\begin{aligned} (D^\mu \mathcal{C}_n[R])(\gamma) &= 0, \\ \forall \gamma \in \pi\Gamma' \text{ and } \forall \mu \in \mathbb{N}_0^n \text{ with } 0 \leq |\mu| \leq m+l-1. \end{aligned}$$

Since the univariate interpolatory R and the multivariate interpolatory $\mathcal{C}_n[R]$ are connected as follows:

$$R(\omega) = \mathcal{C}_n[R](\omega, 0, \dots, 0), \quad \forall \omega \in \mathbb{T},$$

we have $(D^k R)(\pi) = D^{(k, 0, \dots, 0)} \mathcal{C}_n[R](\pi, 0, \dots, 0) = 0$ for all $0 \leq k \leq m+l-1$. Hence the accuracy number of R is at least $m+l$, which contradicts to the given assumption. Therefore the accuracy number of $\mathcal{C}_n[R]$ has to be m .

B. Proof of Theorem 2

In this subsection we prove Theorem 2. In the proof we use the concepts of Compression-Alignment-Prediction (CAP) and Compression-Alignment-Modified-Prediction (CAMP) [33]. CAMP is a variant of CAP, and CAP is a generalization of the Laplacian pyramid [32]. In particular, CAP without alignment operator is the same as Laplacian pyramid. It is well known that Laplacian pyramid has a trivial reconstruction algorithm of reversing the steps in its decomposition algorithm. Both CAP and CAMP are originally designed for the redundant wavelet construction, and CAMP is introduced in order to achieve a better space localization than CAP.

Given $\tau := \mathcal{C}_n[S]$, $\tau^d := \mathcal{C}_n[U]$ with interpolatory U , and $t_\nu(\omega) := e^{-i\nu\cdot\omega} \overline{U(\omega \cdot \nu + \pi)}$, $\omega \in \mathbb{T}^n$, $\nu \in \Gamma'$, we want to show that there exist dual wavelet masks $t_\nu^d(\omega)$ such that $(\tau, (t_\nu)_{\nu \in \Gamma'})$ and $(\tau^d, (t_\nu^d)_{\nu \in \Gamma'})$ satisfy the MUEP conditions in (3).

To show this, first let us construct another pair of wavelet masks $(\tau_\nu)_{\nu \in \Gamma}$ and dual wavelet masks $(\tau_\nu^d)_{\nu \in \Gamma}$, which we know for sure satisfy the MUEP conditions with τ and τ^d .

First extend the definition of t_ν by defining t_0 :

$$t_\nu(\omega) := \begin{cases} \frac{1}{2}(1 - \tau(\omega)), & \text{if } \nu = 0, \\ e^{-i\nu\cdot\omega} \overline{U(\omega \cdot \nu + \pi)}, & \text{if } \nu \in \Gamma'. \end{cases}$$

Then from [33] we know that

$$t_\nu(\omega) = 2^{\frac{n}{2}-1} \cdot t_{-\nu}^{CAMP}(\omega), \quad \nu \in \Gamma, \quad (19)$$

where t_ν^{CAMP} is the CAMplet mask in Section 2.3 of [33].

Furthermore by comparing the CAPlet masks in Lemma 2.2 of [33] with the CAMplet masks, it is easy to see that they are related as

$$t_\nu^{CAP}(\omega) - t_\nu^{CAMP}(\omega) = \begin{cases} 0, & \text{if } \nu = 0, \\ f_{-\nu}(\omega) t_0^{CAMP}(\omega), & \text{if } \nu \in \Gamma', \end{cases} \quad (20)$$

where $f_\nu(\omega) = e^{-i\nu\cdot\omega} \sum_{\gamma \in \pi\Gamma} e^{-i\nu\cdot\gamma} \overline{\tau^d(\omega + \gamma)}$. Here it is necessary to point out that f_ν is π -periodic, i.e. $f_\nu(\omega + \gamma) = f_\nu(\omega)$, for any $\gamma \in \pi\Gamma$.

Now define $(\tau_\nu)_{\nu \in \Gamma}$

$$\tau_\nu(\omega) := 2^{\frac{n}{2}-1} t_{-\nu}^{CAP}(\omega), \quad \nu \in \Gamma. \quad (21)$$

Then since CAP without alignment operator is the same as Laplacian pyramid, and Laplacian pyramid has a trivial reconstruction, we know that with

$$\tau_\nu^d(\omega) := \begin{cases} 2^{1-n}, & \text{if } \nu = 0, \\ 2^{1-n} e^{-i\nu\cdot\omega}, & \text{if } \nu \in \Gamma', \end{cases}$$

$(\tau, (\tau_\nu)_{\nu \in \Gamma})$ and $(\tau^d, (\tau_\nu^d)_{\nu \in \Gamma})$ satisfy the MUEP conditions.

Next, we start from the MUEP conditions of $(\tau, (\tau_\nu)_{\nu \in \Gamma})$ and $(\tau^d, (\tau_\nu^d)_{\nu \in \Gamma})$ to find our dual wavelet masks t_ν^d . To do that, we need three more identities. The first one is a simple observation that can be obtained from (19), (20) and (21):

$$\tau_\nu(\omega) - t_\nu(\omega) = \begin{cases} 0, & \text{if } \nu = 0, \\ f_\nu(\omega) t_0(\omega), & \text{if } \nu \in \Gamma'. \end{cases} \quad (22)$$

The second one can be derived from the interpolatory property of τ^d and the identities (17):

$$\tau_0^d(\omega) + \sum_{\nu \in \Gamma'} \overline{f_\nu(\omega)} \tau_\nu^d(\omega) = 2\tau^d(\omega). \quad (23)$$

After defining $g_\nu(\omega) := e^{-i\nu\cdot\omega} \sum_{\gamma \in \pi\Gamma} e^{-i\nu\cdot\gamma} \overline{\tau(\omega + \gamma)}$, the third identity:

$$t_0(\omega) + 2^{-n} \sum_{\nu \in \Gamma'} t_\nu(\omega) \overline{g_\nu(\omega)} = 0 \quad (24)$$

can be shown from the biorthogonality between τ and τ^d and the identities (17). Finally from the above identities (22), (23) and (24), with

$$\delta_{\gamma 0} := \begin{cases} 1, & \text{if } \gamma = 0, \\ 0, & \text{if } \gamma \in \pi\Gamma', \end{cases}$$

we get

$$\begin{aligned} & \delta_{\gamma 0} \\ &= \overline{\tau(\omega + \gamma)} \tau^d(\omega) + \overline{\tau_0(\omega + \gamma)} \tau_0^d(\omega) + \sum_{\nu \in \Gamma'} \overline{\tau_\nu(\omega + \gamma)} \tau_\nu^d(\omega) \\ &= \overline{\tau(\omega + \gamma)} \tau^d(\omega) + \overline{t_0(\omega + \gamma)} \tau_0^d(\omega) \\ & \quad + \sum_{\nu \in \Gamma'} \overline{f_\nu(\omega + \gamma) t_0(\omega + \gamma) + t_\nu(\omega + \gamma)} \tau_\nu^d(\omega) \\ &= \overline{\tau(\omega + \gamma)} \tau^d(\omega) \\ & \quad + \overline{t_0(\omega + \gamma)} \left(\tau_0^d(\omega) + \sum_{\nu \in \Gamma'} \overline{f_\nu(\omega + \gamma)} \tau_\nu^d(\omega) \right) \\ & \quad + \sum_{\nu \in \Gamma'} \overline{t_\nu(\omega + \gamma)} \tau_\nu^d(\omega) \\ &= \overline{\tau(\omega + \gamma)} \tau^d(\omega) + \overline{t_0(\omega + \gamma)} 2\tau^d(\omega) + \sum_{\nu \in \Gamma'} \overline{t_\nu(\omega + \gamma)} \tau_\nu^d(\omega) \\ &= \overline{\tau(\omega + \gamma)} \tau^d(\omega) \\ & \quad + \left(\overline{t_0(\omega + \gamma)} + 2^{-n} \sum_{\nu \in \Gamma'} \overline{t_\nu(\omega + \gamma)} g_\nu(\omega + \gamma) \right) 2\tau^d(\omega) \\ & \quad - 2^{-n} \sum_{\nu \in \Gamma'} \overline{t_\nu(\omega + \gamma)} g_\nu(\omega + \gamma) 2\tau^d(\omega) \\ & \quad + \sum_{\nu \in \Gamma'} \overline{t_\nu(\omega + \gamma)} \tau_\nu^d(\omega) \\ &= \overline{\tau(\omega + \gamma)} \tau^d(\omega) - 2^{1-n} \sum_{\nu \in \Gamma'} \overline{t_\nu(\omega + \gamma)} g_\nu(\omega + \gamma) \tau^d(\omega) \\ & \quad + \sum_{\nu \in \Gamma'} \overline{t_\nu(\omega + \gamma)} \tau_\nu^d(\omega) \\ &= \overline{\tau(\omega + \gamma)} \tau^d(\omega) \\ & \quad + \sum_{\nu \in \Gamma'} \overline{t_\nu(\omega + \gamma)} (-2^{1-n} g_\nu(\omega) \tau^d(\omega) + \tau_\nu^d(\omega)). \end{aligned}$$

Therefore, by letting $t_\nu^d := -2^{1-n} g_\nu \tau^d + \tau_\nu^d$, we find the dual wavelet masks t_ν^d , $\nu \in \Gamma'$, such that $(\tau, (t_\nu)_{\nu \in \Gamma'})$ and $(\tau^d, (t_\nu^d)_{\nu \in \Gamma'})$ satisfy the MUEP conditions.

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